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Monotone Quadratic Spline Interpolation

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1. INTRODUCTION

Let $X = \{x_i\}$, $Y = \{y_i\}$, $-\infty < i < \infty$; $x_{i-1} < x_i$ and $y_{i-1} < y_i$ for all i. Let $S_n = S_n(X)$ be the set of splines of degree n, having knots at x_i, $-\infty < i < \infty$; i.e., each $f \in S_n$ coincides with a polynomial of degree $\leqslant n$ on (x_{i-1}, x_i), $-\infty < i < \infty$, and $f \in C^{n-1}$. The problem of monotone spline interpolation is: *What additional conditions on the sequences X and Y will guarantee the existence of an increasing* $f \in S_n$ *satisfying* $f(x_i) = y_i$, $-\infty < i < \infty$?

The similarity in character of this problem to that of cardinal spline interpolation, as studied by Schoenberg [1, 2], is apparent by considering the following typical example of the latter [2]: Let $F_s = \{F(x): F(x) \in C(-\infty, \infty), F(x) = 0(|x|^s) \text{ as } x \to \pm\infty\}$. What conditions on Y will guarantee the existence of a function $f \in S_n \cap F_s$, satisfying $f(i) = y_i$, $-\infty < i < \infty$? (Note that in this case the interpolation nodes are the integers, while we consider more general nodes.)

The problem of monotone spline interpolation is trivial for S_1 (piecewise linear functions), so we turn our attention to quadratic splines. We obtain sufficient conditions for the existence of an increasing quadratic spline interpolant; we also consider the problem of convex interpolation.

2. THE MAIN RESULTS

Let $s_i = (y_i - y_{i-1})/(x_i - x_{i-1})$, $-\infty < i < \infty$.

THEOREM 1. *Suppose* $y_i - y_{i-1} > 0$, $s_i - s_{i-1} > 0$ *for all i. Then there exists a unique increasing function* $f \in S_2$, *such that* $f(x_i) = y_i$ *for all i. If, in addition,* $s_i - 2s_{i-1} + s_{i-2} \geqslant 0$ *for all i, then f is convex.*

Remark. Note that the conditions $s_i - s_{i-1} > 0$ imply that the data are convex, in the sense that the slopes increase from left to right. The additional conditions, $s_i - 2s_{i-1} + s_{i-2} \geqslant 0$, guarantee that f is also convex.

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Proof. The proof is based on the fact that if $f(x) = ax^2 + bx + c$ has a nonnegative derivative at α and at $\beta > \alpha$, then f is nondecreasing on $[\alpha, \beta]$.

Since $\{s_i\}$ is a monotone and nonnegative sequence, there exists $\gamma \geqslant 0$ such that $\lim_{n\to-\infty} s_n = \gamma$. Assume, initially, that $\gamma = 0$.

Let f reduce to p_n on $[x_{n-1}, x_n]$, where $p_n(x) = a_n x^2 + b_n x + c_n$, $-\infty < n < \infty$. Then p_1 must satisfy:

$$p_1(x_0) = x_0^2 a_1 + x_0 b_1 + c_1 = y_0, \tag{1}$$

$$p_1(x_1) = x_1^2 a_1 + x_1 b_1 + c_1 = y_1, \tag{2}$$

$$p_1'(x_0) = 2x_0 a_1 + b_1 \geqslant 0, \tag{3}$$

$$p_1'(x_1) = 2x_1 a_1 + b_1 \geqslant 0. \tag{4}$$

Eliminating c_1 from (1) and (2), we obtain $b_1 = s_1 - (x_0 + x_1) a_1$. Substituting in (3) and (4), we obtain two conditions that a_1 must satisfy,

$$-s_1/(x_1 - x_0) \leqslant a_1 \leqslant s_1/(x_1 - x_0). \tag{5}$$

For $n > 1$, p_n must satisfy:

$$p_n(x_{n-1}) = x_{n-1}^2 a_n + x_{n-1} b_n + c_n = y_{n-1} \tag{6}$$

$$p_n(x_n) = x_n^2 a_n + x_n b_n + c_n = y_n, \tag{7}$$

$$p_n'(x_{n-1}) = 2x_{n-1} a_n + b_n = 2x_{n-1} a_{n-1} + b_{n-1} = p_{n-1}'(x_{n-1}), \tag{8}$$

$$p_n'(x_n) = 2x_n a_n + b_n \geqslant 0. \tag{9}$$

From (6), (7), and (8) we obtain

$$\begin{aligned} a_n &= \frac{1}{(x_n - x_{n-1})} [s_n - (2x_{n-1} a_{n-1} + b_{n-1})], \\ b_n &= \frac{1}{(x_n - x_{n-1})} [(x_n + x_{n-1})(2x_{n-1} a_{n-1} + b_{n-1}) - 2x_{n-1} s_n]. \end{aligned} \tag{10}$$

Thus,

$$2x_n a_n + b_n = 2s_n - (2x_{n-1} a_{n-1} + b_{n-1}). \tag{11}$$

Using (11) recursively, we obtain

$$\begin{aligned} 2x_n a_n + b_n = {} & 2(s_n - s_{n-1} + \cdots + (-1)^n s_2) \\ & + (-1)^{n+1} s_1 + (-1)^{n+1} (x_1 - x_0) a_1 . \end{aligned} \tag{12}$$

Let $\alpha_0 = \alpha_1 = s_1$ and $\alpha_n = 2(s_n - s_{n-1} + \cdots + (-1)^n s_2) + (-1)^{n+1} s_1$, $n = 2, 3, \ldots$. Then, by (5), (9), and (12), a_1 must satisfy

$$\begin{aligned} a_1 &\leqslant \alpha_n/(x_1 - x_0), && n \text{ even}, \\ a_1 &\geqslant -\alpha_n/(x_1 - x_0), && n \text{ odd}. \end{aligned} \tag{13}$$

Note that $\alpha_n = \alpha_{n-2} + 2(s_n - s_{n-1}) \geqslant \alpha_{n-2}$. Thus $0 \leqslant \alpha_0 \leqslant \alpha_2 \leqslant \cdots$ and $0 \leqslant \alpha_1 \leqslant \alpha_3 \leqslant \cdots$. Hence, $\cdots \leqslant -\alpha_3 \leqslant -\alpha_1 \leqslant 0 \leqslant \alpha_0 \leqslant \alpha_2 \leqslant \cdots$, so that there certainly exists a_1 satisfying (13).

For $n < 1$, p_n must satisfy:

$$p_n(x_{n-1}) = x_{n-1}^2 a_n + x_{n-1} b_n + c_n = y_{n-1}, \tag{14}$$

$$p_n(x_n) = x_n{}^2 a_n + x_n b_n + c_n = y_n, \tag{15}$$

$$p_n{}'(x_n) = 2x_n a_n + b_n = 2x_n a_{n+1} + b_{n+1} = p'_{n+1}(x_n), \tag{16}$$

$$p_n{}'(x_{n-1}) = 2x_{n-1} a_n + b_n \geqslant 0. \tag{17}$$

From (14), (15), and (16), we obtain

$$\begin{aligned} a_n &= \frac{1}{(x_n - x_{n-1})} [(2x_n a_{n+1} + b_{n+1}) - s_n] \\ b_n &= \frac{1}{(x_n - x_{n-1})} [2x_n s_n - (x_n + x_{n-1})(2x_n a_{n+1} + b_{n+1})]. \end{aligned} \tag{18}$$

Thus,

$$2x_{n-1} a_n + b_n = 2s_n - (2x_n a_{n+1} + b_{n+1}). \tag{19}$$

We now use (19) recursively to obtain

$$\begin{aligned} 2x_{n-1} a_n + b_n = 2(s_n - s_{n+1} + \cdots + (-1)^n s_0) \\ + (-1)^{n+1} s_1 + (-1)^n (x_1 - x_0) a_1 . \end{aligned} \tag{20}$$

Let $\beta_n = 2(s_n - s_{n+1} + \cdots + (-1)^n s_0) + (-1)^{n+1} s_1$, $n = 0, -1, -2\ldots.$ Then, by (17) and (20), a_1 must also satisfy

$$\begin{aligned} a_1 &\leqslant \beta_n/(x_1 - x_0), \qquad n \text{ odd}, \\ a_1 &\geqslant -\beta_n/(x_1 - x_0), \qquad n \text{ even}. \end{aligned} \tag{21}$$

From the hypotheses of the theorem, $2 \sum_{k=0}^{\infty} (-1)^k s_{-k}$ converges to a nonnegative number, t. Let

$$a_1 = (s_1 - t)/(x_1 - x_0). \tag{22}$$

Then $s_1 - t \geqslant -s_1$, since $2s_1 - t = 2 \sum_{k=-1}^{\infty} (-1)^{k+1} s_{-k} \geqslant 0$. Thus $-\alpha_1 = -s_1 \leqslant (x_1 - x_0) a_1 \leqslant s_1 = \alpha_0$, so that inequalities (13) are satisfied. Also,

$$\begin{aligned} (x_1 - x_0) a_1 &= s_1 - 2(s_0 - s_{-1} + \cdots + s_{2n}) + 2(s_{2n-1} - s_{2n-2} + \cdots) \\ &= -\beta_{2n} + 2(s_{2n-1} - s_{2n-2} + \cdots), \end{aligned} \tag{23}$$

and, similarly,

$$(x_1 - x_0) a_1 = \beta_{2n+1} - 2(s_{2n} - s_{2n-1} + \cdots). \tag{24}$$

Hence, inequalities (21) are also satisfied, so that conditions (1)–(4), (6)–(9) and (14)–(17) can be fulfilled, and thus a monotone interpolating quadratic spline exists.

To prove uniqueness, notice that $\lim_{n\to-\infty} -\beta_{2n} = \lim_{n\to-\infty} \beta_{2n+1} = s_1 - t$. Since a_1 must satisfy (21) for all n, the choice $a_1 = (s_1 - t)/(x_1 - x_0)$ is the only one possible.

Suppose now that $s_i - 2s_{i-1} + s_{i-2} \geqslant 0$ for all i. From (10), (11), (18), (19), and (22), we obtain $a_n = s_n - 2s_{n-1} + 2s_{n-2} - 2s_{n-3} + \cdots = (s_n - 2s_{n-1} + s_{n-2}) + (s_{n-2} - 2s_{n-3} + s_{n-4}) + \cdots$. Since each term in parentheses is $\geqslant 0$, we have that $a_n \geqslant 0$ for all n. Hence f is convex, and the theorem is proved in the case $\gamma = 0$.

Now suppose $\gamma > 0$. Let $g(x) = \gamma x$ and $\bar{y}_i = y_i - g(x_i)$. Then $\bar{y}_i - \bar{y}_{i-1} = (y_i - y_{i-1}) - \gamma(x_i - x_{i-1}) > 0$ since $s_i \geqslant \gamma$. Also, $\bar{s}_i - \bar{s}_{i-1} = s_i - s_{i-1} > 0$ and $\lim_{n\to-\infty} \bar{s}_n = 0$, so that we can apply the first part of the theorem. Let $\bar{f} \in S_2$ be the monotone interpolant for the data $(x_i, \bar{y}_i)$. Then $f(x) = \bar{f}(x) + \gamma x$ is a monotone interpolant for the data (x_i, y_i), and $f \in S_2$. Moreover, if $\bar{f}$ is convex, then so is f.

Remark. The conditions $s_i - s_{i-1} > 0$ are not necessary. Indeed, other sufficient conditions are $D\ s_i - s_{i-1} < 0$ for all i. But the theorem may hold without either of these conditions. For example, let $s_n = 1$ for $n \leqslant 1$, $s_2 = 2$, $s_3 = 4$, and $s_n = 3$ for $n \geqslant 4$. Then the data are neither convex nor concave, but (13) and (21) will be satisfied if $-1 \leqslant (x_1 - x_0)\, a_1 \leqslant 1$. On the other hand, if $s_1 = 1$, $s_2 = 4$, $s_3 = 2$, then the data are neither convex nor concave, and (13) cannot be satisfied.

If we consider finite interpolation, at nodes $\{x_i\}_0^k$, or semi-infinite interpolation, at nodes $\{x_i\}_0^\infty$, then the interpolating spline will no longer be unique, since a_1 need only satisfy inequalities (13), which are implied by inequalities (5). Thus any value of a_1 satisfying (5) will yield a monotone interpolating quadratic spline. If we specify the value of $f'(x_0)$, then f will be unique. However, we do not have complete freedom in the choice of $f'(x_0)$.

THEOREM 2. *Suppose* $y_i - y_{i-1} > 0$, $i = 1, 2,\ldots,$ *and* $s_i - s_{i-1} > 0$, $i = 2, 3,\ldots$. *Let* y_0' *satisfy* $0 \leqslant y_0' \leqslant 2s_1$. *Then there exists a unique monotone function* $f \in S_2$, *such that* $f(x_i) = y_i$, $i = 0, 1, 2,\ldots,$ *and* $f'(x_0) = y_0'$. *Furthermore, if* $0 \leqslant y_0' \leqslant s_1$, $y_0' \geqslant 2s_1 - s_2$ *and* $s_i - 2s_{i-1} + s_{i-2} \geqslant 0$, $i = 3, 4, 5,\ldots,$ *then* f *is convex.*

Proof. $f'(x_0) = p_1'(x_0) = 2x_0 a_1 + b_1 = s_1 - (x_1 - x_0)\, a_1$. Since $-s_1 \leqslant (x_1 - x_0)\, a_1 \leqslant s_1$, by (5), we have that $0 \leqslant p_1'(x_0) \leqslant 2s_1$. Thus, if $0 \leqslant y_0' \leqslant 2s_1$, and if we let $a_1 = (s_1 - y_0')/(x_1 - x_0)$, then f will be uniquely defined and will satisfy $f'(x_0) = y_0'$. If $0 \leqslant y_0' \leqslant s_1$, then $a_1 \geqslant 0$. Also, from (10), $a_2 = (1/(x_2 - x_1))(s_2 - s_1 - (x_1 - x_0)a_1)$. Since $(x_1 - x_0)\, a_1 = s_1 - y_0'$, a_2 will be $\geqslant 0$ if $y_0' \geqslant 2s_1 - s_2$. Again, from (10),

$$a_n = \frac{1}{(x_n - x_{n-1})} (s_n - 2s_{n-1} + \cdots + (-1)^n\, 2s_2 + (-1)^{n+1}\, s_1 + (-1)^{n+1}(x_1 - x_0)\, a_1), \qquad \text{for} \quad n \geqslant 3.$$

Hence, if $s_i - 2s_{i-1} + s_{i-2} \geqslant 0$, $i = 3, 4, 5, \ldots$, then $a_n \geqslant 0$ for $n \geqslant 3$, so that f is convex.

References

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2. I. J. Schoenberg, Cardinal interpolation and spline functions. II. Interpolation of data of power growth, *J. Approximation Theory* **6** (1972), 404–420.